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Hierarchical Decentralized Stabilization for Networked Dynamical Systems by LQR Selective Pole Shift

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Abstract: This paper proposes a systematic method to design hierarchical, decentralized, stabilizing controllers for homogeneous hierarchical dynamical networks. Based on LQR approach with a properly chosen performance index including global and local objectives with control input penalty, an obtained optimal LQR feedback gain gives the closed-loop system a prescribed desirable hierarchical structure. In addition, the undesirable eigenvalues of the given homogeneous network can be selectively shifted by further selecting the weighting matrices based on the left eigenvectors associated with those eigenvalues. Finally, the proposed method is summarized into a systematic design procedure with an illustrative numerical example to show its effectiveness.

1. INTRODUCTION

Many systems in the real world such as world wide web, power grids, social networks and gene networks are hierarchical dynamical systems. Since the hierarchical structure provides an ability to achieve a global target in the network by designing the local communication topologies, a large network design problem can be significantly reduced to smaller ones. Moreover, the network can be designed in a decentralized manner which is usually required in many practical systems. Because of those advantages, hierarchical dynamical systems have been extensively investigated in a variety of research fields including control.

In Williams et al. [2004], the formation of a vehicle network is shown to be stable if the graphs representing the communication structures in the layers are properly selected. In another study Smith et al. [2005], a hierarchical network is developed for increasing the rate of consensus among vehicles but the structure in each layer is limited to be cyclic. Recently, Hamilton and Broucke [2010] introduced a concept of patterned linear systems for a restrictive class of structures. Since the hierarchical networks in the real world usually have dense interactions inside the subnetworks and sparse communication between them Fortunato [2010], the proposed frameworks in those researches failed to describe this characteristic.

Motivated by that fact, Shimizu and Hara [2008, 2009], Hara et al. [2009] generalized the hierarchical cyclic pursuit scheme and emphasized the effect of low rank inter-layer interactions by aggregating and distributing information in the network to achieve the rapid consensus. Subsequently, Tsubakino and Hara [2012], Fujimori et al. [2011] continued this line of research and presented a new class of low rank intergroup connection namely eigen-connection to analyze and design hierarchical networks such that only some specific eigenvalues of the local interconnection matrices are selectively shifted.

Nevertheless, many existing results on hierarchical networked control so far are for the analysis and only a few works deal with systematic synthesis. On the other hand, to develop a systematic procedure for control system design, one of natural ways is to set up an LQR optimal control problem. Hence, there are several works investigating the mechanism of preserving certain desirable hierarchical structures in the LQR framework (e.g. Motee et al. [2008]). Tsubakino et al. [2013] generalized the results in Motee et al. [2008] and introduced more general classes of structured matrices that preserve their structures under the LQR setting. Borrelli and Keviczky [2008] investigated identical decoupled systems and proposed a way to design sub-optimal controllers based on the LQR approach. Massioni and Verhaegen [2009] studied a class of two-layer hierarchical networks that is similar to the one in Tsubakino and Hara [2012] and proposed an optimization-based approach to design distributed controllers for such type of networks.

The purpose of this paper is to propose a new, systematic method to design hierarchical decentralized optimal controllers for homogeneous dynamical networks based on LQR approach with notion of low-rank inter-layer interactions examined in Tsubakino and Hara [2012], Fujimori et al. [2011]. The main focus in this new method is how to choose the weighting matrices in the LQR setting to derive a stabilizing controller (or state feedback gain) which has a prescribed desirable hierarchical structure. Accordingly, we propose a systematic way of doing it by considering a class of performance indexes consisting of both global (or upper layer) and local (or lower layer) objectives with total control input penalty. Therefore, the desired structure of the network can be achieved by choosing appropriate weighting matrices in the performance index.

To this end, we introduce an idea of selective pole shift proposed in Kawasaki and Shimemura [1983], Kawasaki et al. [1989], Kraus and Kucera [1999], Cigler and Kucera [2009] for non-hierarchical systems. We then try to link the eigen-connection properties required in Tsubakino and Hara [2012] with the idea of selective pole shift from the view point of hierarchical decentralized control synthesis.

One of advantages of our LQR-based synthesis is to take the global and local objectives into account in addition to just stabilizing by decentralized control. Another contribution of the proposed method is the ability to selectively shifted the unexpected eigenvalues in the network without affecting to other eigenvalues.

The paper is organized as follows. In Section 2, we explain the goal of this paper after introducing the model of two-layer homogeneous hierarchical networks. We then set the performance index to be minimized in the LQR setting, which has a certain hierarchical structure and show the structure of the optimal feedback gain in Section 3. Section 3.2 provides a systematic design procedure for our purpose with an illustrative example to show its effectiveness. The detailed procedure of selective pole shift is explained in Section 4. Finally, some remarks are given in Section 5.

2. PROBLEM FORMULATION

2.1 Hierarchical Networked Dynamical Systems

Consider a homogeneous multi-agent dynamical system having N agents in which the mathematical model of each agent is as follows,

$$\begin{aligned} \dot{x}_i &= A_1 x_i + B_1 u_i, \\ y_i &= C_1 x_i, i = 1, \ldots, N, \end{aligned} \tag{1}$$

with $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $0 < m \le n_1$, $C_1 \in \mathbb{R}^{p \times n_1}$, where $x_i \in \mathbb{R}^{n_1}$ is the state vector of the ith subsystem, and $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$ are the vectors containing all the inputs and measured outputs of the ith subsystem, respectively. Denote $P(s)$ the transfer function of the subsystems, i.e.,

$$P(s) = C_1(sI_{n_1} - A_1)^{-1}B_1.$$

Hence, the initial hierarchical network without controller can be represented by

$$\begin{aligned} \dot{x} &= \mathcal{A}_1 x + \mathcal{B}_1 u, \\ y &= \mathcal{C}_1 x, \end{aligned} \tag{2}$$

where $\mathcal{A}_1 = I_N \otimes A_1, \mathcal{B}_1 = I_N \otimes B_1, \mathcal{C}_1 = I_{n_2} \otimes C_1$, $x = [x_1^T, \ldots, x_N^T]^T$, $u = [u_1^T, \ldots, u_N^T]^T$, $y = [y_1^T, \ldots, y_N^T]^T$. We here assume that all the states of agents are measurable, i.e., $C_1 = I_{n_1}$ and hence $y_i = x_i$. In the case where only partial state variables are measurable, we can design a stabilizing hierarchical output feedback controller by combining a completely local observer which estimates the local state and the obtained state feedback controller. Due to space limitation, we do not present it in this paper.

The information exchange in the real multi-agent systems controlled with a decentralized fashion is as follows: (i) Each agent sends out a unique aggregated signal to collaborate with other connected agents to realize the global objectives in addition to the local objectives. (ii) Simultaneously, each agent is able to receive the signals sent by other connected agents individually.

Let us denote $\mathcal{G}$ the graph representing the information structure in a multi-agent system, where each node in $\mathcal{G}$ stands for an agent and each edge in $\mathcal{G}$ represents the interconnection between two agents. In this paper, we assume that the communications between agents are bidirectional, i.e., $\mathcal{G}$ is undirected. Then, the information structure in a multi-agent system can be mathematically characterized by a matrix K, where the elements of K stands for the weights on the edges of $\mathcal{G}$, or equivalently the weights for the information exchanges between agents. Denote $\mathcal{E}$ the edge set of $\mathcal{G}$, then the class of K is defined by

$$\mathbb{K}_s := \{K = K^T \in \mathbb{R}^{N \times N} \mid K_{ij} = 0 \text{ if } (i,j) \notin \mathcal{E}\}. \tag{3}$$

From the theoretical point of view, such a multi-agent system can be considered as a two-layer hierarchical system, where each agent is cast as a subsystem in the lower layer and those subsystems are interconnected in the upper layer. Accordingly, the interconnection among agents in a multi-agent system can be treated in the associated two-layer hierarchical system as follows. The ith subsystem tries to collaborate with all other subsystems by sending a unique aggregated signal $z_i \in \mathbb{R}^m$, receiving a partial set of aggregated signals $z_j \in \mathbb{R}^m$ from the jth subsystem satisfying $(i,j) \in \mathcal{E}$, and determining a kind of reference command $w_i \in \mathbb{R}^m$ for the global objectives in the simplest way as

$$w_i = \sum_{(i,j)\in\mathcal{E}} K_{ij} z_j, \; i = 1, \ldots, N. \tag{4}$$

Moreover, we also allow each subsystem to be implemented with a local controller whose output is denoted by $u_{\ell,i}, i = 1, \ldots, N$. Hence, the control input for each subsystem has the following form

$$u_i = w_i + u_{\ell,i}. \tag{5}$$

As a result, the control input for the whole hierarchical network is represented by

$$u = w + u_\ell = (K \otimes I_m)z + u_\ell, \tag{6}$$

where $w = \begin{bmatrix} w_1^T & \cdots & w_N^T \end{bmatrix}^T$, $z = \begin{bmatrix} z_1^T & \cdots & z_N^T \end{bmatrix}^T$, $u_\ell = \begin{bmatrix} u_{\ell,1}^T & \cdots & u_{\ell,N}^T \end{bmatrix}^T$. Subsequently, Figure 1 describes the whole hierarchical dynamical networked control system, where the interaction among subsystems is represented by the term $K \otimes I_m$, $G(s)$ denotes the transfer function of the locally controlled subsystems (agents).

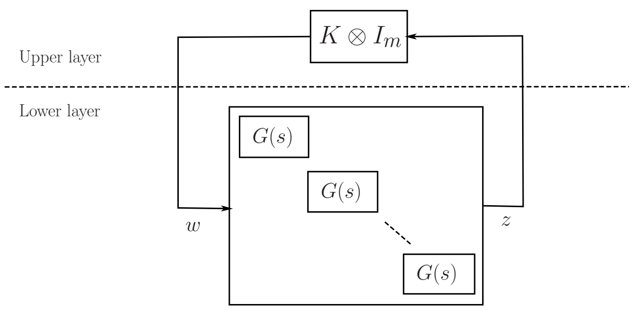

Fig. 1. Block diagram of hierarchical networked control system.

The question here is how to design $z_i, u_{\ell,i}$ $(i = 1, \ldots, N)$ and $K \in \mathbb{K}_s$ in a systematic way to achieve both the global and local objectives as well as the stabilization of the whole networked system. This is actually our hierarchical decentralized controller design, which will be explained in the next subsection.

2.2 Hierarchical Decentralized Design Problem

Figure 2 shows the structure of the locally controlled subsystems (agents) in the lower layer where the subsystems (agents) are controlled by local, state feedback controllers.

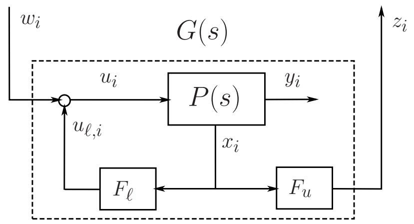

Fig. 2. Block diagram of state feedback case.

Consequently, the state feedback design problem is stated as follows.

State feedback design problem: For the given network $\mathcal{G}$ with (A_1, B_1) controllable, the design problem is to determine the higher level interconnection gain $K \in \mathbb{K}_s$ and the lower layer state feedback gains $F_\ell \in \mathbb{R}^{m\times n_1}$ and $F_u \in \mathbb{R}^{m\times n_1}$, which respectively produce the local feedback signal $u_{\ell,i} = F_\ell x_i$ and the aggregated signal to be sent $z_i = F_u x_i$ for $i = 1, \ldots, N$.

The reference command (4) of the ith subsystem and the control input (6) yields a state feedback form for $\dot{x} = \mathcal{A}_1 x$ as

$$u = \mathcal{F}x,$$

where $\mathcal{F}$ belongs to the following class:

$$\mathbb{F}_K := \{\mathcal{F} \in \mathbb{R}^{Nm\times Nn_1} \mid \mathcal{F} = I_N \otimes F_\ell + K \otimes F_u\}. \quad (7)$$

The first term $I_N \otimes F_\ell$ in the expression of $\mathcal{F}$ is associated with the local feedback signals $u_{\ell,i}, i = 1, \ldots, N$ while the second term $K \otimes F_u$ represents the interactions among subsystems through the aggregated signals $z_i, i = 1, \ldots, N$ since

$$(K \otimes F_u)x = [(K \otimes I_m)(I_N \otimes F_u)]x,$$

Consequently, the sate feedback design problem is reduced to determine F_ℓ, F_u and $K \in \mathbb{K}_s$. In order to do this systematically, we will propose a procedure based on the LQR (Linear Quadratic Regulator) design, which can take the global/local objectives into account, in the next two sections.

3. HIERARCHICAL STATE FEEDBACK LQR DESIGN

3.1 Class of Performance Indexes

We here introduce the design of a state feedback hierarchical decentralized LQR controller for the network (2) with the assumption that all states of agents are measurable.

Consider the following performance indexes

$$J = J_{x,\mathcal{L}} + J_{x,\mathcal{G}} + J_u, \quad (8)$$

where

$$\begin{aligned} J_{x,\mathcal{L}} &= \int_0^\infty x^T(I_{n_2} \otimes Q_\ell)x\, dt, \\ J_{x,\mathcal{G}} &= \int_0^\infty x^T(K \otimes Q_g)x\, dt, \\ J_u &= \int_0^\infty u^T R u\, dt, \end{aligned} \quad (9)$$

where $Q_\ell \in \mathbb{R}^{n_1\times n_1}, Q_g \in \mathbb{R}^{n_1\times n_1}$, $Q_\ell \succeq 0, Q_g \succeq 0$; $\mathcal{R} \in \mathbb{R}^{Nm\times Nm}$, $\mathcal{R} \succ 0$; $K \in \mathbb{K}_s^+$ which is the class of positive semidefinite interconnection defined as follows,

$$\mathbb{K}_s^+ := \{K \in \mathbb{K}_s \mid K \text{ is positive semidefinite}\}. \quad (10)$$

$J_{x,\mathcal{L}}$ is a local performance index composing of the individual penalties for the states of subsystems. $J_{x,\mathcal{G}}$ corresponds to a global performance index taking into account the interconnections among subsystems represented by matrix K. J_u is a penalty for the control input required to the whole network.

The global performance index $J_{x,\mathcal{G}}$ is employed to improve the control performance. Of course, the subsystems can be stabilized by themselves independently without introducing $J_{x,\mathcal{G}}$. However we may have a better control performance in the presence of $J_{x,\mathcal{G}}$. For instance, the convergence of the agents' states to zero will be faster. The matrix K here corresponds to the communication structure in the network, i.e., $K_{ij} = K_{ji} \neq 0$ if the ith subsystem and the jth subsystem are connected and $K_{ij} = 0$ if the ith subsystem and the jth subsystem are unconnected. Hence, the information structure of the network is taken into account in the global performance index $J_{x,\mathcal{G}}$. Moreover, the elements of K as well as matrix Q_g put some weights on the relative states of subsystems leading to the improvement on the convergence of subsystems' states.

Subsequently, rewriting the performance index (8) as follows,

$$J = \int_0^\infty (x^T \mathcal{Q} x + u^T \mathcal{R} u)dt, \quad (11)$$

where

$$\mathcal{Q} = I_N \otimes Q_\ell + K \otimes Q_g, \quad (12)$$

we aim at designing a hierarchical decentralized optimal LQR controller for the given hierarchical network (2) which minimizes the performance index (8).

Employing the following assumptions:

A1: $(\mathcal{A}, \mathcal{B})$ is controllable,
A2: $(\mathcal{Q}^{1/2}, \mathcal{A})$ is observable,

it is shown from the optimal control theory Anderson and Moore [1990] that such an LQR controller is computed by $u = \mathcal{F}x$, $\mathcal{F} \in \mathbb{R}^{(mN)\times(n_1N)}$ where

$$\mathcal{F} = -\mathcal{R}^{-1}\mathcal{B}^T\mathcal{P},$$

with $\mathcal{P} \in \mathbb{R}^{(n_1N)\times(n_1N)}$ is the unique positive definite solution of the following Riccati equation

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} + \mathcal{Q} - \mathcal{P}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^T\mathcal{P} = 0. \quad (13)$$

The assumption **A1** is actually equivalent to the controllability of (A_1, B_1). Motivated by the class of hierarchical decentralized feedback gains (7) and the structure of the weighting matrix $\mathcal{Q}$, we select the weighting matrix $\mathcal{R}$ with the following form

$$\mathcal{R}^{-1} = I_N \otimes R_\ell + K \otimes R_g, \tag{14}$$

where $R_\ell \in \mathbb{R}^{m\times m}, R_g \in \mathbb{R}^{m\times m}$, $R_\ell \succ 0, R_g \succ 0$.

In the previous works, it was proved that if $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{R}, \mathcal{Q}$ belong to some operator algebra Motee et al. [2008] or semigroup Tsubakino et al. [2013] then the solution $\mathcal{P}$ of the Riccati equation (13) also belongs to that algebra or semigroup. As a result, they could prove that the LQR controller gain $\mathcal{F}$ has a similar property. However, in our current setting, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{R}, \mathcal{Q}$ do not belong to any operator algebra or semigroup. Therefore, it is not possible to show that with the choice of the weighting matrices as in (12) and (14), $\mathcal{P}$ has the same structure.

Therefore, in the next subsection, we will propose another way of choosing the weighting matrices $\mathcal{Q}$ and $\mathcal{R}$ of the forms of (12) and (14), respectively, which completely fits our situation and purpose.

3.2 Design Procedure

In this subsection, we propose a systematic design procedure for state feedback hierarchical decentralized controllers which consists of four steps. Employing this design procedure, we obtain a hierarchical feedback controller which fulfills the two design requirements in the state feedback design problem.

Here are the steps of the design procedure.

- **Step 1 (Local LQR Design) :**
  Select the weighting matrices for the local objectives, $Q_\ell \in \mathbb{R}^{n_1\times n_1}$ and $R_\ell \in \mathbb{R}^{m\times m}$ such that $(Q_\ell^{1/2}, A_1)$ is observable and $R_\ell \succ 0$, and solve the corresponding local Riccati equations
  $$P_\ell A_1 + A_1^T P_\ell - P_\ell B_1 R_\ell B_1^T P_\ell + Q_\ell = 0. \tag{15}$$
  to obtain the unique positive definite solution $P_\ell \in \mathbb{R}^{n_1\times n_1}$.
- **Step 2 (Setting Upper Layer Interactions) :**
  Choose a positive semidefinite matrix $K \in \mathbb{K}_s^+$.
- **Step 3 (Global LQR Setting) :**
  Choose $R_g \in \mathbb{R}^{m\times m}, R_g \succ 0$ and set $Q_g \in \mathbb{R}^{n_1\times n_1}$ as follows:
  $$Q_g = P_\ell B_1 R_g B_1^T P_\ell,$$
  where $P_\ell \in \mathbb{R}^{n_1\times n_1}$ is the solution of the local Riccati equation (15).
- **Step 4 (State Feedback Gain Calculation) :**
  Set the state feedback gains F_ℓ and F_u as follows:
  $$\begin{aligned} F_\ell &= -R_\ell B_1^T P_\ell, \\ F_u &= -R_g B_1^T P_\ell. \end{aligned}$$

The following theorem clearly shows the validation of the above procedure in which the resultant LQR controller belong to the class $\mathbb{F}_K$ in (7) if the weighting matrices are chosen as in the design procedure.

Theorem 1. Consider a set of subsystems represented by (1) with (A_1, B_1) controllable for all $i = 1, \ldots, N$. Let K be a matrix in class $\mathbb{K}_s^+$ and the weighting matrices $\mathcal{Q}$ and $\mathcal{R}$ have the forms (12) and (14) with $R_g \in \mathbb{R}^{m\times m}$ and $Q_g \in \mathbb{R}^{n_1\times n_1}$ chosen as in Step 3 of the state feedback design procedure in Subsection 3.2. Then the optimal hierarchical LQR state feedback gain is given by

$$F = I_N \otimes F_\ell + K \otimes F_u, \tag{16}$$

where

$$F_\ell = -R_\ell B_1^T P_\ell, F_u = -R_g B_1^T P_\ell. \tag{17}$$

Proof Assumption **A1** is obvious since it is equivalent to the controllability of $(A_1, B_1) \forall i = 1, \ldots, N$. On the other hand, Assumption **A2** holds when $Q_g = 0$. With extra term $K \otimes Q_g$ in $\mathcal{Q}$, which is positive semidefinitethe observability condition is not broken, and hence Assumption **A2** holds even for any $Q_g \succeq 0$. In addition, $\mathcal{R}^{-1} = I_N \otimes R_\ell + K \otimes R_g \succ 0$, i.e., $\mathcal{R} \succ 0$ since $R_\ell \succ 0$. Thus, (13) has a unique positive definite solution.

Next, substituting $\mathcal{P} = I_N \odot P_\ell$ and $\mathcal{Q}, \mathcal{R}$ back to the Riccati equation (13), we obtain

$$\begin{aligned} 0 = & I_N \otimes (P_\ell A_1 + A_1^T P_\ell - P_\ell B_1 R_\ell B_1^T P_\ell + Q_\ell) \\ & + K \otimes (Q_g - P_\ell B_1 R_g B_1^T P_\ell). \end{aligned} \tag{18}$$

This is always true with $Q_g = P_\ell B_1 R_g B_1^T P_\ell$ and P_ℓ is the solution of (15). Hence, $\mathcal{P} = I_N \odot P_\ell$ is a solution of (13). Since we have assumed the uniqueness of the solution of (13), that matrix $\mathcal{P}$ is the unique solution. Accordingly, the LQR controller is calculated as follows,

$$\begin{aligned} \mathcal{F} &= -\mathcal{R}^{-1}\mathcal{B}^T\mathcal{P}, \\ &= -(I_N \otimes R_\ell + K \otimes R_g)(I_N \otimes B_1^T)(I_N \otimes P_\ell), \\ &= -I_N \otimes (R_\ell B_1^T P_\ell) - K \otimes (R_g B_1^T P_\ell). \end{aligned}$$

Thus, the LQR controller gains F_ℓ and F_u are determined by (16).

3.3 Illustrative example

Consider a network of 3 identical subsystems where each subsystem is described by (1) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{19}$$

Then let K be a Laplacian matrix as follows,

$$K = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+q & -q \\ 0 & -q & q \end{bmatrix}, q \geq 0. \tag{20}$$

This matrix K implies that the 1st and the 2nd subsystems are connected, the 2nd and the 3rd subsystems may be connected while the 1st and the 3rd subsystems are not connected. Subsequently, we can rewrite the global performance index as follows,

$$J_{x,\mathcal{G}} = (x_1 - x_2)^T Q_g (x_1 - x_2) + q(x_2 - x_3)^T Q_g (x_2 - x_3). \tag{21}$$

It then can be seen that $J_{x,\mathcal{G}}$ puts a penalty under the form of a quadratic function with a weight matrix Q_2 for the difference between the states of subsystems and hence by minimizing J including $J_{x,\mathcal{G}}$, the gaps between the states of subsystems are reduced simultaneously with the decrease of the states. As a result, the convergence speed of the agents' states to zero will be faster as both $J_{x,\mathcal{L}}$ and $J_{x,\mathcal{G}}$ are utilized than when only $J_{x,\mathcal{L}}$ is used. Furthermore, by changing the value of q, the system responses are also changed.

The simulation result in Figure 3 exhibits the output responses of subsystems without a global peformance index, in which the output of each subsystem is equal to the first state in that subsystem and when a global performance index is employed with $q = 0$, i.e., the 2nd and 3rd subsystems are not connected. Obviously, when the global performance index is employed but $q = 0$, the output of the 1st subsystem converge to the output of the 2nd subsystem before all the outputs of subsystems come to zero. This is because only those subsystems are connected while the 3rd subsystem is not connected to any of them.

Next, Figure 4 shows the output responses of subsystems when $q = 10$ and $q = 20$. We can observe that the convergence speed in these two cases are faster than in the last cases. Furthermore, the output of the 2nd subsystem rapidly converge to the output of the 3rd subsystem before all the outputs of subsystems come to zero as q increase. This is because a much larger weight is put on the state difference of the 2nd and 3rd subsystems making them converge to each other faster. In other words, by letting q larger the network is divided into two groups of subsystems in which the first group include the 1st subsystem and the second group composes of the 2nd and 3rd subsystems. Thus, the structure of the network is clearly reflected in the interconnection matrix K and changing the elements of K can improve the control performance.

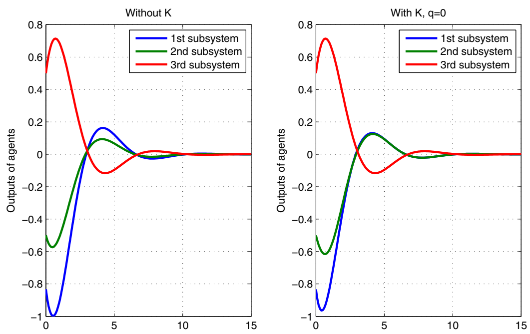

Fig. 3. System responses without (left) and with (right) a global performance index but $q = 0$.

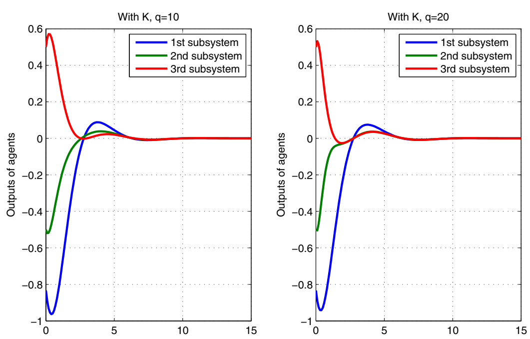

Fig. 4. System responses with a global performance index as $q = 10$ (left) and $q = 20$ (right).

4. SELECTIVE POLE SHIFT BY DECENTRALIZED LQR CONTROLLER

For simplicity, we only consider the situation that the local interconnection matrix A_1 contains two undesirable eigenvalues. This leads to a fact that there are $2N$ repeated undesirable eigenvalues in the whole network. Then the proposed approach can be extended similarly to more general contexts.

Suppose that (λ_1, λ_2) are undesirable eigenvalues of A_1 and (ν_1^*, ν_2^*) are the associated left-eigenvectors. We select a sub-weighting matrix Q_1 of the form

$$Q_1 = [\nu_1 \ \nu_2]\, \mathcal{Q}_1 \begin{bmatrix} \nu_1^* \\ \nu_2^* \end{bmatrix} \succeq 0, \tag{22}$$

then the solution P_1 of the Riccati equation (15) also has the form

$$P_1 = [\nu_1 \ \nu_2]\, \mathcal{P}_1 \begin{bmatrix} \nu_1^* \\ \nu_2^* \end{bmatrix} \tag{23}$$

with a positive definite matrix $\mathcal{P}_1$. Replacing Q_1 and P_1 back to (15), we obtain

$$\mathcal{P}_1\Gamma + \Gamma^*\mathcal{P}_1 - \mathcal{P}_1\mathcal{R}_1\mathcal{P}_1 + \mathcal{Q}_1 = 0, \tag{24}$$

where

$$\begin{aligned} \Gamma &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \\ \mathcal{R}_1 &= \begin{bmatrix} \nu_1^* \\ \nu_2^* \end{bmatrix} B_1 R_1 B_1^T \, [\nu_1 \ \nu_2] \,. \end{aligned} \tag{25}$$

Solving the 2×2 Riccati equation (24) gives us matrix $\mathcal{P}_1$ and then P_1 is calculated from (23). Clearly, $\text{rank}(P_1) = 2$ which is in general small in comparison with the number of agents n_1 in each subsystem.

Denote

$$E = B_1^T \, [\nu_1 \ \nu_2] \,, E \in \mathbb{R}^{m \times 2}. \tag{26}$$

Then the following theorem shows the selective pole shift through the eigenvalue distribution of the closed-loop interconnection matrix A.

Theorem 2. With the LQR feedback controller $u = Fx$ defined in Theorem 1 and the sub-weighting matrix Q_1 selected in (22), the eigenvalue set the closed-loop interconnection matrix A is determined as follows,

$$\sigma(A) = \left(\bigcup_{\gamma \in \sigma(K)} \sigma(\Xi_\gamma) \right) \bigcup \left(\sigma(A_1) \backslash \{\lambda_1, \lambda_2\} \right), \tag{27}$$

where Ξ_γ is a 2×2 matrix defined by

$$\Xi_\gamma = \Gamma - E^*(R_1 + \gamma R_2) E \mathcal{P}_1, \tag{28}$$

and E is defined in (26).

Proof Due to the limitation of space, we ignore the proof here.

Thanks to Theorem 2, only undesirable eigenvalues are selectively moved. Furthermore, we can determine the eigenvalue spectrum of the closed-loop interconnection matrix A based on the eigenvalues of the local interconnection matrix A_1 and the matrix Ξ_γ. Although the eigenvalues of Ξ_γ are not analytically obtained but Ξ_γ is a 2×2 matrix, so its eigenvalues can be easily calculated.

Consider a network of 50 agents which is divided into 10 groups and each group has 5 agents. Suppose that the system matrices in each group are given by

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad (29)$$

The eigenvalues of A_1 are $1.2249 \pm 0.8516i, -0.8716 \pm 1.3358i, -0.7065$ of which $1.2249 \pm 0.8516i$ are unstable. Consequently, the whole network has two unstable eigenvalues, each one has the multiplicity equals to 10.

Suppose that all states of agents are measurable and we can freely choose but fix the interconnection matrix between the subsystems. Now, we apply our proposed design procedure to design a hierarchical decentralized controller such that the network's hierarchical structure is preserved and the unstable eigenvalues are properly shifted. Let $\lambda_1 = 1.2249 - 0.8516i, \lambda_2 = 1.2249 + 0.8516i$ then the left eigenvectors of A_1 corresponding to λ_1 and λ_2 can be easily computed. Next, we choose $\mathcal{Q}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $R_1 = 1$, then we can compute $\mathcal{R}_1, \mathcal{P}_1$ and P_1. Let $R_2 = 1$ and K be a Laplacian matrix which satisfies the positive semidefiniteness in Theorem 1.

Then the eigenvalues of the closed-loop interconnection matrix A and the states of all agents are exhibited in Figure 5. We can see that all eigenvalues of the closed-loop system belong to the left-haft complex plane and that only unstables eigenvalues are changed while the other stable eigenvalues remain unaltered. As a result, the states of all agents converge to 0 as observed in the subplot on the right hand side.

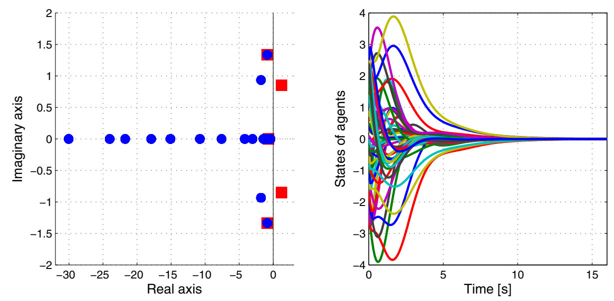

Fig. 5. (Left) ■ : Old eigenvalues, • : New eigenvalues; (Right) States of agents in the designed network.

5. CONCLUSION

We have presented in this paper a systematic method to design hierarchical, decentralized LQR controllers for homogeneous hierarchical dynamical networks. The important features of the proposed controller are as follows. Firstly, it preserves the expected hierarchical structure of network. Secondly, it is able to selectively shift the undesirable eigenvalues in the network and keep other eigenvalues unchanged. Lastly, the proposed controller is globally optimal.

An extension of the current work is to design hierarchical, decentralized controllers for heterogeneous hierarchical networks. Another direction is to design feedback controllers for hierarchical networks such that all subsystems reach a consensus instead of stabilization. We will present the results for those extensions in the forthcoming papers.

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