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A new necessary and sufficient stability condition for linear time-delay systems

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Abstract: This paper deals with linear time-invariant systems with multiple delays. We present a necessary and sufficient stability condition, expressed in terms of the delay Lyapunov matrix. This result generalizes the well known Lyapunov theorem for delay free linear systems. An illustrative example shows how to apply the presented condition.

Keywords: time delay systems; Lyapunov-Krasovskii functionals; Lyapunov matrix; stability conditions.

1. INTRODUCTION

It is well known that the exponential stability of a linear time-invariant system of ordinary differential equations is equivalent to the positive-definiteness of a solution of the corresponding Lyapunov matrix equation. This paper is devoted to an extension of this result to the case of time-delay systems.

The idea to use Lyapunov type functionals instead of Lyapunov functions for stability analysis of time-delay systems was proposed in Krasovskii (1956). In Repin (1965) an approach to compute functionals with prescribed time derivatives was given. Then, the approach has been studied in Infante and Castelan (1978), Huang (1989), Louisell (1998). In Kharitonov and Zhabko (2003), a class of complete type functionals was introduced. The functionals of the class admit a quadratic lower bound if the corresponding system is exponentially stable. In some sense these functionals can be considered as a generalization of the quadratic Lyapunov functions, usually used for ordinary differential equations. It is worth mentioning that a complete type functional is defined by an auxiliary Lyapunov matrix, which in this case is matrix-valued.

Some aspects related with the computation of Lyapunov matrices have been studied in Mondié et al. (2011), Jarlebring et al. (2011), Kharitonov (2013). The complete type functionals have been applied for robust stability analysis in Kharitonov and Zhabko (2003), for obtaining exponential estimates of solutions of time-delay systems in Kharitonov and Hinrichsen (2004), for the computation of the norm of the transfer matrix in Jarlebring et al. (2011), and for some other purposes, see Kharitonov (2013). However, we believe that the direct application of the complete type functionals to the stability analysis of linear systems have not received due attention. A criteria of the exponential stability of a scalar single delay equation have been given in Mondié (2012) and in Egorov and Mondié (Vestnik, 2013). Some new sufficient stability conditions based on Lyapunov functionals of complete type were given in Medvedeva and Zhabko (2013).

Recently, in Egorov and Mondié (RNC, 2013) and Egorov and Mondié (TDS, 2013), the complete type functionals have been used in order to derive an extensive family of necessary stability conditions for linear systems with multiple delays. These conditions are based exclusively on the corresponding Lyapunov matrix. It has been demonstrated that these conditions can be effectively applied to derive exact stability regions for time-delay systems with parameters. The main contribution of the present paper is to show that it is possible to single out among these necessary conditions some that are the sufficient ones. In other words, a necessary and sufficient condition for the exponential stability of a linear system with multiple delays is obtained.

Sections 2 and 3 are devoted to some preliminary results, whereas the main result with its proof is given in Section 4. The paper ends with an illustrative example and conclusions.

2. PRELIMINARIES

In this paper we consider a linear delay system of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j), \quad t \geqslant 0, \tag{1}$$

where $A_0, \ldots, A_m$ are constant real $n \times n$ matrices, delays are ordered as follows: $0 = h_0 < h_1 < \ldots < h_m = H$.

Assume that the initial function φ is piecewise continuous, $\varphi \in PC([-H, 0], \mathbb{R}^n)$, i.e., it has a finite number of discontinuity points of the first kind. The restriction of the solution $x(t, \varphi)$ of system (1) on the interval $[t - H, t]$ is denoted by

$$x_t(\varphi) : \theta \to x(t + \theta, \varphi), \quad \theta \in [-H, 0].$$

The Euclidian norm for vectors is denoted $\| \cdot \|$. For functions we use the seminorm $\| \cdot \|_{\mathcal{H}}$:

$$\|\varphi\|_{\mathcal{H}} = \sqrt{\|\varphi(0)\|^2 + \int_{-H}^{0} \|\varphi(\theta)\|^2 \, d\theta}.$$

Definition 1. System (1) is said to be *exponentially stable*, if there exist constants $\gamma \geqslant 1$, $\sigma > 0$, such that

$$\|x(t,\varphi)\| \leqslant \gamma e^{-\sigma t}\|\varphi\|_{\mathcal{H}}, \quad t \geqslant 0.$$

A matrix-valued function $K(t)$, which is solution of the equation

$$\dot{K}(t) = \sum_{j=0}^{m} A_j K(t-h_j), \quad t \geqslant 0, \tag{2}$$

with the initial conditions

$$K(0) = I, \quad K(t) = 0,\ t < 0, \tag{3}$$

is called *the fundamental matrix* of system (1). This matrix satisfies also (see Bellman and Cooke (1963), p.180) the equation

$$\dot{K}(t) = \sum_{j=0}^{m} K(t-h_j)A_j, \quad t \geqslant 0.$$

The notation $Q > 0$ means that the symmetric matrix Q is positive definite. And $Q \ngeqslant 0$ means that the matrix Q is not positive semidefinite, i.e., there exists a vector μ, such that $\mu^T Q \mu < 0$. The square block matrix with i-th row and j-th column element A_{ij} is denoted $\{A_{ij}\}_{i,j=1}^{r}$.

3. LYAPUNOV-KRASOVSKII FRAMEWORK

A functional, which satisfies the equality

$$\frac{dv_0(x_t(\varphi))}{dt} = -x^T(t,\varphi)Wx(t,\varphi)$$

for a fixed positive definite matrix W, was presented in Kharitonov and Zhabko (2003). It has the form

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) \\ &+ 2\varphi^T(0)\sum_{j=1}^{m}\int_{-h_j}^{0} U^T(\theta+h_j)A_j\varphi(\theta)\,d\theta \\ &+ \sum_{k=1}^{m}\int_{-h_k}^{0}\varphi^T(\theta_1)A_k^T\left(\sum_{j=1}^{m}\int_{-h_j}^{0} U(\theta_1+h_k-\theta_2-h_j)\right. \\ &\left. \times A_j\varphi(\theta_2)\,d\theta_2\right)d\theta_1. \end{aligned}$$

Each term of the sum contains the matrix-valued function $U(\tau)$, which is named *the delay Lyapunov matrix, associated with matrix W*. It satisfies the set of equations

$$U'(\tau) = \sum_{j=0}^{m} U(\tau-h_j)A_j, \quad \tau \geqslant 0, \tag{4}$$

$$U(\tau) = U^T(-\tau), \quad \tau \geqslant 0, \tag{5}$$

$$\sum_{j=0}^{m}\left[U(-h_j)A_j + A_j^T U(h_j)\right] = -W. \tag{6}$$

Condition (4) is called *dynamic property*, condition (5) — *symmetric property*, and (6) — *algebraic property.*

Theorem 2. (Kharitonov (2013)). The delay Lyapunov matrix of system (1), associated with a given symmetric matrix W, exists and is unique if and only if *the Lyapunov condition* holds, i.e., there are no eigenvalues s_1, s_2 of the system, such that $s_1 + s_2 = 0$.

In Kharitonov and Zhabko (2003), the so-called *complete type functionals*, which admit a quadratic lower bound when the system is exponentially stable, were introduced:

$$\begin{aligned} v(\varphi) &= v_0(\varphi) \\ &+ \sum_{j=1}^{m}\int_{-h_j}^{0}\varphi^T(\theta)\left[W_j + (h_j+\theta)W_{m+j}\right]\varphi(\theta)\,d\theta, \end{aligned}$$

where $W_0, W_1, \ldots, W_{2m}$ are positive definite matrices. In Egorov and Mondié (TDS, 2013), a more simple functional

$$v_1(\varphi) = v_0(\varphi) + \int_{-H}^{0}\varphi^T(\theta)W\varphi(\theta)\,d\theta$$

was introduced. The derivative of this functional along the solutions of system (1) is equal to

$$\frac{dv_1(x_t(\varphi))}{dt} = -x^T(t-H,\varphi)Wx(t-H,\varphi).$$

Theorem 3. (Egorov and Mondié (TDS, 2013)). If system (1) is exponentially stable, then there exists $\alpha > 0$, such that

$$v_1(\varphi) \geqslant \alpha\|\varphi\|_{\mathcal{H}}^2, \quad \varphi \in PC([-H,0],\mathbb{R}^n).$$

4. MAIN RESULT

Let us introduce the block matrix

$$\mathcal{K}_r(\tau_1,\ldots,\tau_r) = \Big\{U(-\tau_i+\tau_j)\Big\}_{i,j=1}^{r}, \tag{7}$$

which depends exclusively on the delay Lyapunov matrix. In the case of equidistant points

$$\tau_i = \frac{i-1}{r-1}H, \quad i = \overline{1,r},$$

this matrix is of the form

$$\mathcal{K}_r\left(0,\frac{1}{r-1}H,\ldots,\frac{r-2}{r-1}H,\,H\right) = \left\{U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r}.$$

The main result of this contribution, a stability criterion for linear time-delay systems, follows.

Theorem 4. System (1) is exponentially stable if and only if the Lyapunov condition holds and for every natural number $r \geqslant 2$

$$\left\{U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r} > 0. \tag{8}$$

Moreover, if the Lyapunov condition holds and system (1) is unstable, then there exists a natural number r, such that

$$\left\{U\left(\frac{j-i}{r-1}H\right)\right\}_{i,j=1}^{r} \ngeqslant 0.$$

Remark 5. For $r = 2$, condition (8) takes the form

$$\begin{pmatrix} U(0) & U(H) \\ U^T(H) & U(0) \end{pmatrix} > 0, \tag{9}$$

for $r = 3$,

$$\begin{pmatrix} U(0) & U(H/2) & U(H) \\ U^T(H/2) & U(0) & U(H/2) \\ U^T(H) & U^T(H/2) & U(0) \end{pmatrix} > 0, \tag{10}$$

for $r = 4$,

$$\begin{pmatrix} U(0) & U(H/3) & U(2H/3) & U(H) \\ U^T(H/3) & U(0) & U(H/3) & U(2H/3) \\ U^T(2H/3) & U^T(H/3) & U(0) & U(H/3) \\ U^T(H) & U^T(2H/3) & U^T(H/3) & U(0) \end{pmatrix} > 0, \tag{11}$$

and so on.

Remark 6. In contrast with systems of ordinary differential equations, time-delay systems are infinite-dimensional. This explains the fact that in order to determine the stability of such systems, a countable number of inequalities have to be verified.

4.1 Proof of the main result. Necessity

For exponentially stable systems, the Lyapunov condition holds. It remains to prove that inequality (8) holds for every natural r. Actually, it is a direct consequence of the main result of paper Egorov and Mondié (TDS, 2013).

Define the function

$$\bar{\varphi}(\theta) = \sum_{i=1}^{r} K(\tau_i + \theta)\gamma_i, \tag{12}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_r$ are constant vectors, the points $\tau_1, \tau_2, \ldots, \tau_r \in [0, H]$, and $K(t)$ is the fundamental matrix that was defined in Section 2.

Some new properties of the delay Lyapunov matrix, obtained in Egorov and Mondié (TDS, 2013), were used to deduce the equality

$$v_1(\bar{\varphi}) = \gamma^T \mathcal{K}_r(\tau_1, \ldots, \tau_r)\gamma, \tag{13}$$

where $\tau_1, \tau_2, \ldots, \tau_r \in [0, H]$ and $\gamma = (\gamma_1^T, \gamma_2^T, \ldots, \gamma_r^T)^T$. This equality in conjunction with Theorem 3 led to the following result.

Theorem 7. (Egorov and Mondié (TDS, 2013)). If system (1) is exponentially stable, then

$$\mathcal{K}_r(\tau_1, \ldots, \tau_r) > 0, \tag{14}$$

where $\tau_k \in [0, H]$, $k = \overline{1, r}$, and $\tau_i \neq \tau_j$ if $i \neq j$, and the matrix-valued function $\mathcal{K}_r$ is defined by (7).

In our case the points $\tau_i = \dfrac{i-1}{r-1}H$ belong to the segment $[0, H]$, and $\tau_i \neq \tau_j$ if $i \neq j$. Thus, inequality (8) is just a trivial consequence of Theorem 7.

4.2 Proof of the main result. Sufficiency

Assume that the Lyapunov condition holds, then the delay Lyapunov matrix $U(\tau)$ exists. To prove the sufficiency of Theorem 4, it is enough to show that for unstable system there exists a natural number r, such that

$$\left\{ U\left(\frac{j-i}{r-1}H\right) \right\}_{i,j=1}^{r} \not> 0. \tag{15}$$

This inequality is equivalent to the existence of a vector γ, such that

$$\gamma^T \mathcal{K}_r\left(0, \frac{1}{r-1}H, \ldots, \frac{r-2}{r-1}H, H\right)\gamma < 0. \tag{16}$$

Let us introduce some technical results that are instrumental in our proof. The first result, based on the ideas, introduced in Medvedeva and Zhabko (2013), establishes that if the system is unstable the functional v_1 does not admit a lower bound.

Theorem 8. If system (1) is unstable and satisfies the Lyapunov condition, then for every $\alpha_1 > 0$ there exists a function $\hat{\varphi}$, such that

$$v_1(\hat{\varphi}) \leqslant -\alpha_1. \tag{17}$$

Proof. Integrating the equation

$$\frac{dv_1(x_t)}{dt} = -x^T(t-H)Wx(t-H),$$

we obtain

$$v_1(x_T(\varphi)) - v_1(\varphi) = -\int_{-H}^{T-H} x^T(t,\varphi)Wx(t,\varphi)\,dt. \tag{18}$$

Set an arbitrary $\alpha_1 > 0$. System (1) satisfies the Lyapunov condition, so it has no pure imaginary eigenvalues. As the system is unstable, there exists at least one eigenvalue $\lambda = \alpha + i\beta$ with positive real part. Hence, there are two vectors C_1, C_2, at least one of which is nonzero, such that

$$\tilde{x}(t) = e^{\alpha t}\left(\cos\beta t\, C_1 + \sin\beta t\, C_2\right)$$

is a solution of system (1), which is obviously not identically zero on $[-H, T-H]$ for any $T > 0$. Therefore, $\int_{-H}^{T-H} \|\tilde{x}(t)\|^2\,dt \neq 0$. The function $\bar{x}(t) = a\tilde{x}(t)$ is also a solution of the system, corresponding to the initial function $\hat{\varphi}(\theta) = a\tilde{x}(\theta)$, $\theta \in [-H, 0]$. Here a is an arbitrary constant.

If $\beta \neq 0$, then set $T = 2\pi/|\beta|$, while for $\beta = 0$ set $T = 1$. We have the equality

$$\bar{x}(T+\theta) = e^{\alpha T}\hat{\varphi}(\theta), \quad \theta \in [-H, 0].$$

The functional $v_1(\varphi)$ is a quadratic one, so

$$v_1(\bar{x}_T(\hat{\varphi})) = e^{2\alpha T} v_1(\hat{\varphi}).$$

From equality (18) we obtain:

$$v_1(\hat{\varphi}) = -\frac{\int_{-H}^{T-H} \bar{x}^T(t,\bar{\varphi})W\bar{x}(t,\bar{\varphi})\,dt}{e^{2\alpha T} - 1} \leqslant -a^2 \frac{\lambda_{\min}(W)\int_{-H}^{T-H} \|\tilde{x}(t)\|^2\,dt}{e^{2\alpha T} - 1}.$$

Finally, inequality (17) follows by choosing

$$a = \sqrt{\frac{\alpha_1\left(e^{2\alpha T} - 1\right)}{\lambda_{\min}(W)\int_{-H}^{T-H} \|\tilde{x}(t)\|^2\,dt}}.$$

□

Introduce now the class of functions of the form (12) with equidistant points τ_i that we used in the proof of necessity:

$$F([-H,0],\mathbb{R}^n) = \left\{ \psi \in PC\left([-H,0],\mathbb{R}^n\right) \;\middle|\; \psi(\theta) = \sum_{i=1}^{r} K(\tau_i+\theta)\gamma_i,\ \ \tau_i = \frac{i-1}{r-1}H,\ \gamma_i \in \mathbb{R}^n \right\}.$$

We present a key result that establishes the fact that it is possible to approximate any continuous function by an element of $F([-H,0],\mathbb{R}^n)$.

Lemma 9. For any $\varphi \in C\left([-H,0],\mathbb{R}^n\right)$ and any $\varepsilon > 0$ there exists a function $\psi \in F\left([-H,0],\mathbb{R}^n\right)$, such that

$$\|\varphi - \psi\|_{\mathcal{H}} < \varepsilon.$$

For the sake of clarity, this technical proof is given in the Appendix.

Applying Theorem 8 and Lemma 9, we now complete the proof of sufficiency of Theorem 4. To this end, suppose that system (1) is unstable. Set an arbitrary number $\alpha_1 > 0$. By Theorem 8, there exists $\hat{\varphi}$, such that $v_1(\hat{\varphi}) \leqslant -\alpha_1$. It is easy to see that functional $v_1(\varphi)$ is continuous at each point, i.e., for any $\alpha_1 > 0$ there exists a number $\Delta > 0$, such that

$$\|\hat{\varphi} - \varphi\|_{\mathcal{H}} < \Delta \quad \Rightarrow \quad |v_1(\hat{\varphi}) - v_1(\varphi)| < \alpha_1.$$

By Lemma 9, there exists $\psi \in F([-H, 0], \mathbb{R}^n)$, such that

$$\|\hat{\varphi} - \psi\|_{\mathcal{H}} < \Delta.$$

By the above mentioned continuity property, $|v_1(\hat{\varphi}) - v_1(\psi)| < \alpha_1$. Therefore,

$$v_1(\psi) < v_1(\hat{\varphi}) + \alpha_1 \leqslant 0.$$

As function ψ has the form

$$\psi(\theta) = \sum_{i=1}^{r} K(\tau_i + \theta)\gamma_i,$$

equality (13) implies that

$$v_1(\psi) = \gamma^T \mathcal{K}_r \left(0, \frac{1}{r-1}H, \ldots, \frac{r-2}{r-1}H, H\right) \gamma < 0.$$

Thus, inequality (16), equivalently (15), is shown, and sufficiency is established.

5. EXAMPLE

The following example illustrates how the presented criterion can be applied to determine the exact stability region for systems with parameters. Consider the time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t-3), \qquad (19)$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -25 & -4 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_2 & 0 \end{pmatrix}.$$

The matrix $U(\tau)$, $\tau \in [0, 3]$, is calculated for $W = I$, using the semianalytical method, introduced in Garcia-Lozano and Kharitonov (2004), which reduces the task of construction of the delay Lyapunov matrix to the boundary value problem for a system of linear ordinary differential equations.

Let us consider the following set of parameters:

$$\{(k_1, k_2) | k_1 \in [-220, 220], \, k_2 \in [-100, 100]\}.$$

We check our stability conditions at points of an equally spaced grid (120 by 120).

First, we take $r = 2$. As mentioned above, in this case condition (8) takes the form (9). The points where this inequality holds are depicted in Figure 1.

Then we increase the number r. For $r = 3$ we have inequality (10), which improves the upper estimate of the exact stability region for system (19), see Figure 2.

We take now $r = 4$, and apply condition (11) for the points of Figure 2. The points where the condition holds are depicted in Figure 3. Further increasing of the number r does not give improvement of the obtained result.

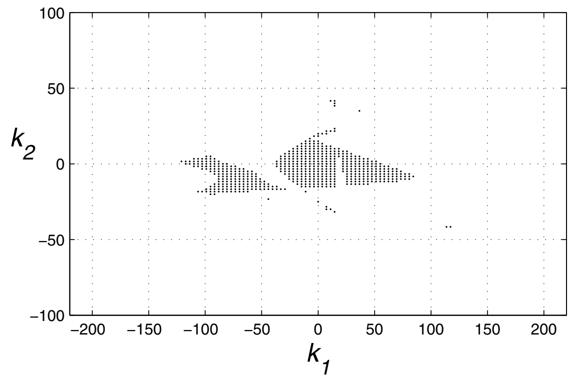

Fig. 1. System (19), inequality (8) for $r = 2$

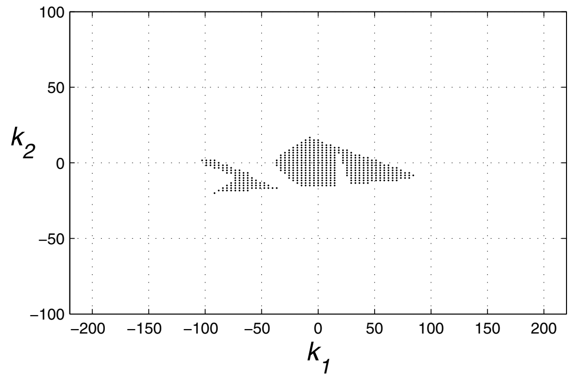

Fig. 2. System (19), inequality (8) for $r = 3$

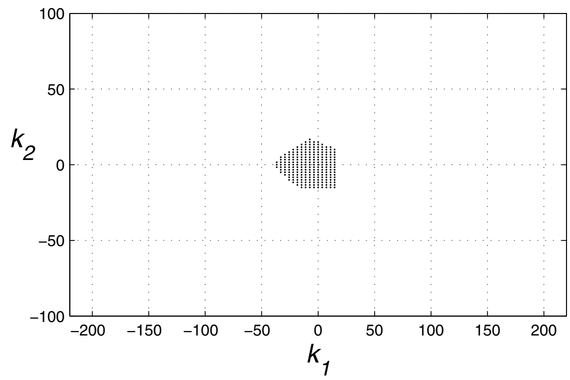

Fig. 3. System (19), inequality (8) for $r = 4, 5, 6, \ldots$

Let us enlarge the domain, depicted in Figure 3, and apply the D-subdivision method, proposed in Neimark (1949). The boundary of the obtained domain coincides with the D-subdivision lines (see Figure 4). Obviously, the marked region is a stable one, as when $k_1 = k_2 = 0$ system (19) reduces to an exponentially stable delay free system. Thus, we have obtained the exact stability region.

To demonstrate the computational complexity associated with the increase of r, we note that the computation time for obtaining Figure 1 is equal to 29 seconds, whereas the time for obtaining Figure 3 is equal to 32 seconds.

Remark 10. It is worth mentioning that for every system with parameters Theorem 4 guarantees the existence of a finite number r, such that inequality (8) gives the exact

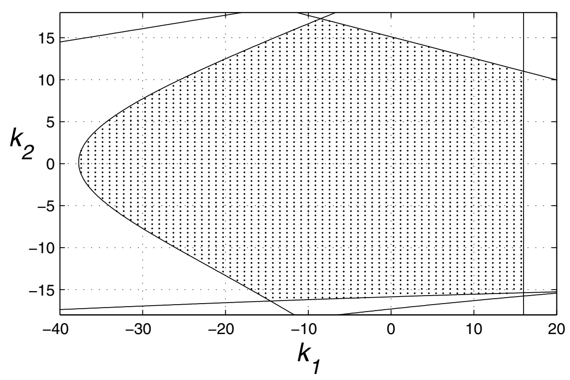

Fig. 4. System (19), the exact stability region

stability region, since we check only a finite number of the parameters' values (in the presented example: grid of 120 by 120 points).

6. CONCLUSION

A necessary and sufficient stability condition, based on the delay Lyapunov matrix, for linear systems with multiple delays is presented. As shown, the obtained criterion can be applied to determine stability regions in the space of parameters.

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Appendix A. PROOF OF LEMMA 9

Given a continuous function φ and a number $\varepsilon > 0$, we have to find a natural number r and vectors $\gamma_1, \ldots, \gamma_r$, such that $\|\varphi - \psi\|_{\mathcal{H}} < \varepsilon$, where

$$\psi(\theta) = \sum_{i=1}^{r} K\left(\frac{i-1}{r-1}H + \theta\right)\gamma_i.$$

The fundamental matrix $K(t)$ has a bounded derivative on $[0, H]$, hence, there exists $L > 0$:

$$\|K(t_1) - K(t_2)\| \leqslant L|t_1 - t_2|, \quad t_1,\, t_2 \in [0, H].$$

Set $M = \|\varphi(-H)\|$ and

$$\varepsilon_1 = \frac{\varepsilon}{\sqrt{H}(1 + LM)e^{LH}}.$$

As any continuous function on a segment is uniformly continuous, there exists a number $\bar{\delta} > 0$, such that for any $\delta \in (0, \bar{\delta})$ the inequality $|t_1 - t_2| \leqslant \delta$ implies $\|\varphi(t_1) - \varphi(t_2)\| < \varepsilon_1$. It is clear that there is δ_1, such that:

$$1)\ \delta_1 \in (0, \bar{\delta}),\ \ 2)\ \delta_1 < \varepsilon_1,\ \ 3)\ \frac{H}{\delta_1} \text{ is a natural number.}$$

For this δ_1 we define

$$r = \frac{H}{\delta_1} + 1 \ \text{ and } \ \tau_k = \frac{k-1}{r-1}H = (k-1)\delta_1,\ k = \overline{1, r}.$$

Let us find γ_k, $k = \overline{1, r}$, from the equalities:

$$\varphi(-\tau_k) = \psi(-\tau_k), \quad k = \overline{1, r}.$$

As $K(t) = 0$ for $t < 0$, and $K(0) = I$,

$$\psi(-\tau_k) = \sum_{i=1}^{r} K(\tau_i - \tau_k)\gamma_i = \sum_{i=k+1}^{r} K(\tau_i - \tau_k)\gamma_i + \gamma_k.$$

Thus,

$$\gamma_r = \varphi(-\tau_r),$$
$$\gamma_k = \varphi(-\tau_k) - \sum_{i=k+1}^{r} K(\tau_i - \tau_k)\gamma_i, \quad k = \overline{1, r-1}.$$

We compute function $\psi(\theta)$. It remains to prove the inequality $\|\varphi - \psi\|_{\mathcal{H}} < \varepsilon$. Consider first $\theta \in [-\tau_r, -\tau_{r-1})$:

$$\begin{aligned}\varphi(\theta) - \psi(\theta) &= \varphi(\theta) - \varphi(-\tau_r) + \psi(-\tau_r) - \psi(\theta) \\ &= \varphi(\theta) - \varphi(-\tau_r) + [K(0) - K(\theta + \tau_r)]\,\gamma_r.\end{aligned}$$

Obviously, $\|\gamma_r\| = \|\varphi(-\tau_r)\| = \|\varphi(-H)\| = M$, therefore,

$$\begin{aligned}\|\varphi(\theta) - \psi(\theta)\| &\leqslant \|\varphi(\theta) - \varphi(-\tau_r)\| \\ &\quad + \|K(0) - K(\theta + \tau_r)\|\,\|\gamma_r\| < \varepsilon_1 + L\delta_1 M.\end{aligned}$$

For brevity we introduce the notation $\varepsilon^{(1)} = \varepsilon_1 + L\delta_1 M$. The fundamental matrix $K(t)$ is continuous for $t \geqslant 0$, therefore,

$$\lim_{\theta \to -\tau_{r-1} - 0} \|\varphi(\theta) - \psi(\theta)\| < \varepsilon^{(1)}.$$

In other words, $\|\varphi(-\tau_{r-1}) - K(\tau_r - \tau_{r-1})\gamma_r\| < \varepsilon^{(1)}$. But

$$\varphi(-\tau_{r-1}) = \psi(-\tau_{r-1}) = K(\tau_r - \tau_{r-1})\gamma_r + \gamma_{r-1}.$$

As a consequence we arrive at the inequality $\|\gamma_{r-1}\| < \varepsilon^{(1)}$.

Similarly, on the next interval $\theta \in [-\tau_{r-1}, -\tau_{r-2})$:

$$\begin{aligned}\varphi(\theta) - \psi(\theta) &= \varphi(\theta) - \varphi(-\tau_{r-1}) + \psi(-\tau_{r-1}) - \psi(\theta) \\ &= \varphi(\theta) - \varphi(-\tau_{r-1}) + [K(\tau_r - \tau_{r-1}) - K(\theta + \tau_r)]\,\gamma_r \\ &\quad + [K(0) - K(\theta + \tau_{r-1})]\,\gamma_{r-1}.\end{aligned}$$

This means that

$$\|\varphi(\theta) - \psi(\theta)\| < \varepsilon_1 + L\delta_1 M + L\delta_1 \varepsilon^{(1)} = \varepsilon^{(2)},$$

where $\varepsilon^{(2)} = \varepsilon^{(1)}(1 + L\delta_1)$. In addition, $\|\gamma_{r-2}\| < \varepsilon^{(2)}$.

By repeating the process, we obtain for $k = \overline{1, r-1}$:

$$\|\gamma_{r-k}\| < \varepsilon^{(k)},$$
$$\|\varphi(\theta) - \psi(\theta)\| < \varepsilon^{(k)}, \quad \theta \in [-\tau_{r-k+1}, -\tau_{r-k}).$$

Here $\varepsilon^{(k)} = \varepsilon^{(k-1)}(1 + L\delta_1)$, $k = \overline{2, r-1}$. Obviously,

$$\begin{aligned}&\varepsilon^{(1)} < \varepsilon^{(2)} < \ldots < \varepsilon^{(r-1)} = \varepsilon^{(1)}(1 + L\delta_1)^{r-2} \\ &= (\varepsilon_1 + L\delta_1 M)(1 + L\delta_1)^{r-2} < \varepsilon_1(1 + LM)\,(1 + L\delta_1)^{r-1} \\ &\qquad = \frac{\varepsilon}{\sqrt{H} e^{LH}} \left(1 + \frac{LH}{r-1}\right)^{r-1}.\end{aligned}$$

It is not difficult to show that

$$\left(1 + \frac{LH}{r-1}\right)^{r-1} \leqslant e^{LH}.$$

Therefore,

$$\|\varphi(\theta) - \psi(\theta)\| < \frac{\varepsilon}{\sqrt{H}}$$

for $\theta \in [-H, 0)$. Hence,

$$\begin{aligned}\|\varphi - \psi\|_{\mathcal{H}} &= \sqrt{\|\varphi(0) - \psi(0)\|^2 + \int_{-H}^{0} \|\varphi(\theta) - \psi(\theta)\|^2\, d\theta} \\ &< \sqrt{\left(\frac{\varepsilon}{\sqrt{H}}\right)^2 \int_{-H}^{0} d\theta} = \frac{\varepsilon}{\sqrt{H}}\sqrt{H} = \varepsilon.\end{aligned}$$

Q.E.D.